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CPV and HFP integrals and their applications in the boundary element method

Subrata Mukherjee*

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, NY 14853, USA

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Abstract

The subject of this paper is Hadamard Finite Part (HFP) and Cauchy Principal Value (CPV) representations of certain singular integrals. Of primary concern here are HFP interpretations of strongly singular integrals for which CPV interpretations also exist. A relationship between the two is stated and proved. This matter is then illustrated by two examples and also discussed in the context of singular boundary integral equations (BIEs) for linear elasticity. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

This paper addresses CPV and HFP integrals in the context of boundary integral equations (BIEs) for linear elasticity. It is concerned with Hadamard Finite Part (HFP) interpretations of strongly singular integrals for which Cauchy Principal Value (CPV) interpretations also exist. A relationship between the two is first stated and proved. This relationship is illustrated by two examples and is then discussed in the context of certain standard and hypersingular BIEs (HBIEs) that are commonly used in boundary element method (BEM) formulations for linear elasticity. *The principal goal of this paper is to help better understand HFP integrals in simple situations in which the CPV versions of the same integrals also exist*. Also, a false statement in an earlier paper (Toh and Mukherjee, 1994) is corrected here.

First some definitions in order to keep this presentation clear. Let m (m = 1,2 or 3) be the dimension of the domain of an integral, and its integrand be of $O(r^{-n})$, as $r = |\mathbf{y} - \mathbf{x}| \to 0$. Here, $\mathbf{x}$ is a source or

* Tel.: +1-607-255-7143; fax: +1-607-255-2011.
E-mail address: sm85@cornell.edu (S. Mukherjee).

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collocation point and $\mathbf{y}$ is a field or integration point. In the interest of simplicity, n is also taken to be a positive integer. Then, standard definitions regarding the nature of the integrals, as shown in Table 1, apply. Also, a one-dimensional (1-D) integral with a logarithmically singular integrand is called weakly singular in this work.

It is well known that (e.g., Krishnasamy et al., 1990; Toh and Mukherjee, 1994; Martin, 1991) some strongly singular integrals (e.g., $\int_{-1}^{1} \mathrm{d}x/x$) admit finite CPVs and others (e.g., $\int_{-1}^{1} \mathrm{d}x/|x|$) do not, while hypersingular integrals (e.g., $\int_{-1}^{1} \mathrm{d}x/x^2$), in general, do not have finite CPVs. It is interesting to note, however, that all the integrals above admit finite HFPs.

The issue of HFP interpretations of strongly singular and hypersingular integrals has been discussed in detail in Toh and Mukherjee (1994). This paper has presented a new interpretation and regularization scheme for such integrals. This interpretation is based on the limit to the boundary (LTB) concept as is clear from Proposition 4.1 of Toh and Mukherjee (1994). Results obtained from this interpretation agree with the results from Krishnasamy et al. (1990) on scattering of acoustic waves and are completely consistent with complex variable formulations for 2-D BIEs (Hui and Mukherjee, 1997). Finally, in a recent paper (Mukherjee and Mukherjee, 1998), this HFP interpretation has been shown to be consistent with a Hypersingular Boundary Contour Method (HBCM) formulation for 3-D linear elasticity.

It is stated in Toh and Mukherjee (1994), however, that the HFP interpretation presented in that paper reduces to the CPV for a strongly singular integral for which the CPV exists. This is not true. In fact, the correct relationship between the two, in such cases, is:

$$I_{\mathrm{HFP}} = I_{\mathrm{CVP}} + \lim{}_{\epsilon \to 0} I_{\epsilon} \tag{1}$$

where I_{HFP} and I_{CPV}, respectively, denote the HFP and CPV values of an integral I, and I_{ϵ} is the value of the same integral on an appropriate "inclusion" or "exclusion" zone around a point at which the integral becomes singular (e.g., see Fig. 1 in Guiggiani et al., 1992). This point can be a regular point on the boundary of a body as well as one that lies on an edge (for 3-D problems) or at a corner (for 2-D or for 3-D problems.)

This paper is organized as follows. A proof of Eq. (1) is first presented for 1-D integrals and then for 2-D integrals in 3-D regions. This is followed by an example of an 1-D integral in a 2-D region that includes collocation at a corner, and a 2-D surface integral in a 3-D region concerning evaluation of solid angles. The solid angle example presents a generalization of a formula that has recently appeared in the literature (Liu, 1998). Finally, the standard (displacement) and hypersingular (stress) BIEs of linear elasticity are considered in this context.

It is important to state the reasons for including these examples in this paper. The first, and obvious reason, of course, is to verify that Eq. (1) remains valid in these situations. Also, perhaps more importantly, these examples illustrate the computation of the HFPs of certain integrals that appear in the BEM, and the relationships of these HFPs with the LTB concept and with the CPVs of these integrals. This author believes that these illustrations are of value to researchers interested in this field.

Table 1
Various types of integrals

$n = 0$	Regular
$n \neq 0 < m$	Weakly singular
$n = m$	Strongly singular
$n = m + 1$	Hypersingular

2. CPV AND HFP

This section presents proofs of Eq. (1), first for 1-D and then for 2-D integrals. This is followed by two examples.

2.1. A proof in one dimension

A proof of Eq. (1), for 1-D CPV integrals, is presented here. Referring to Section 2 in Toh and Mukherjee (1994), let τ:I = [-a,b] $\rightarrow \mathscr{R}$ be a function with a singularity at $x = 0$ of the form $\tau(x) \sim O(1/x)$ and ϕ:I = [-a,b] $\rightarrow \mathscr{R}$ be a regular function of the class $C^{0,\alpha}$ at $x = 0$. One is interested in the integral $\int_{-a}^{b} \tau(x)\phi(x)\mathrm{d}x$. Since, by assumption, its CPV exists, the quantity $\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\tau(x)\mathrm{d}x$ must be finite. Then, according to Eqs. (2–4) of Toh and Mukherjee (1994),

$$\displaystyle{=}\!\!\!\!\!\!\int_{-a}^{b} \tau(x)\phi(x)\,\mathrm{d}x = \int_{-a}^{-\epsilon} \tau(x)\phi(x)\,\mathrm{d}x + \int_{\epsilon}^{b} \tau(x)\phi(x)\,\mathrm{d}x + \int_{-\epsilon}^{\epsilon} \tau(x)\big[\phi(x) - \phi(0)\big]\,\mathrm{d}x + \phi(0)\int_{-\epsilon}^{\epsilon} \tau(x)\,\mathrm{d}x \qquad (2)$$

Canceling the last term on the right hand side of the above equation with the corresponding integral in the previous term, and then taking the limit as $\epsilon \rightarrow 0$, one immediately gets:

$$\displaystyle{=}\!\!\!\!\!\!\int_{-a}^{b} \tau(x)\phi(x)\,\mathrm{d}x = -\!\!\!\!\!\!\int_{-a}^{b} \tau(x)\phi(x)\,\mathrm{d}x + \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \tau(x)\phi(x)\,\mathrm{d}x \qquad (3)$$

This is Eq. (1). Please note that in the above equations $-\!\!\!\!\!\int$ denotes the CPV and $=\!\!\!\!\!\!\int$ the HFP of an integral.

2.2. A proof for 2-D integrals in 3-D regions

A proof of Eq. (1), for 2-D CPV integrals, is presented here. Referring to Section 3 in Toh and Mukherjee (1994), let S be an (open or closed) surface in $\mathscr{R}^3$ and the function $\tau(\mathbf{x},\mathbf{y})$:$S \rightarrow \mathscr{R}$ have its only

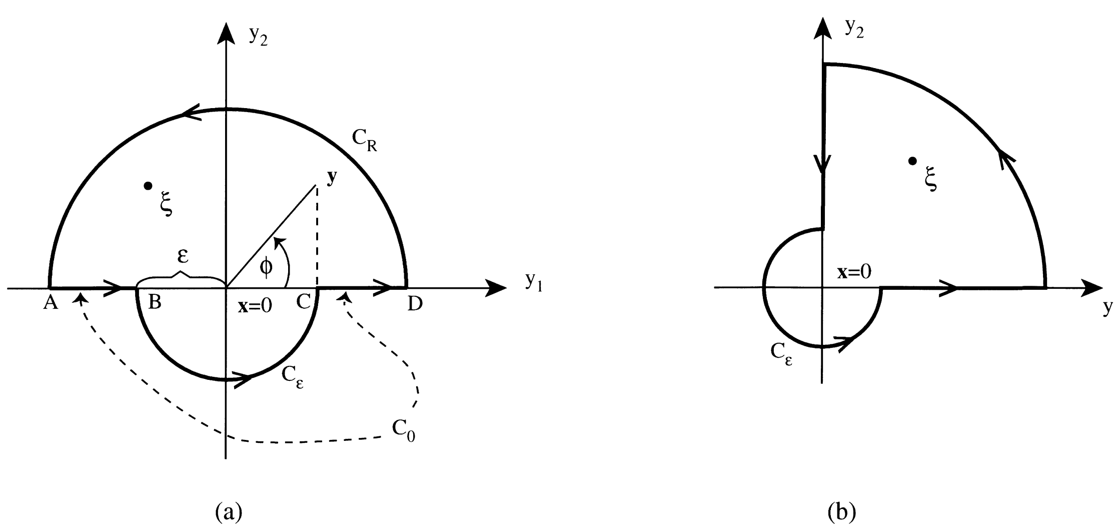

Fig. 1. Geometry of the simple example.

singularity at $\mathbf{y}=\mathbf{x}$ of the form $\tau(\mathbf{x},\mathbf{y}) \sim O(1/r^2)$ with $r=|\mathbf{x}-\mathbf{y}|$. Also, let $\phi : S \rightarrow \mathscr{R}$ be a regular function of the class $C^{0,\alpha}$ at $\mathbf{x} \in S$. Now, let S_ϵ be an inclusion or exclusion neighborhood of $\mathbf{x} \in S$. If S is a closed surface that encloses the body B, S_ϵ is an inclusion neighborhood of $\mathbf{x}$ if the boundary point is approached from inside B, and is an exclusion neighborhood if the boundary point is approached from outside B. This neighborhood S_ϵ is chosen in a symmetric manner consistent with the usual definition of the CPV integral given below. Finally, it is assumed that this CPV integral exists, i.e., $\lim_{\epsilon \to 0} \int_{S_\epsilon} \tau(\mathbf{x},\mathbf{y}) \phi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y})$ is finite.

From Eq. (10) of Toh and Mukherjee (1994), with $\mathbf{x} \in S$, one has:

$$\oint_S \tau(\mathbf{x}, \mathbf{y})\phi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = \langle g_{S_\epsilon}, \phi \rangle + \phi(\mathbf{x}) A(S_\epsilon) = \int_{S \setminus S_\epsilon} \tau(\mathbf{x}, \mathbf{y})\phi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) + \int_{S_\epsilon} \tau(\mathbf{x}, \mathbf{y})[\phi(\mathbf{y}) - \phi(\mathbf{x})] \, \mathrm{d}s(\mathbf{y}) + \phi(\mathbf{x}) \int_{S_\epsilon} \tau(\mathbf{x}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \tag{4}$$

As in the previous proof in one dimension, canceling the last term on the right-hand side of the above equation with the corresponding integral in the previous term, and then taking the limit as $\epsilon \rightarrow 0$, one immediately gets:

$$\oint_S \tau(\mathbf{x}, \mathbf{y})\phi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = ⨍_S \tau(\mathbf{x}, \mathbf{y})\phi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) + \lim_{\epsilon \to 0} \int_{S_\epsilon} \tau(\mathbf{x}, \mathbf{y})\phi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \tag{5}$$

which is Eq. (1).

2.3. An example that involves the angle between two lines

Consider the integral:

$$I = \int_C \frac{\mathrm{d}\phi}{\mathrm{d}s} \, \mathrm{d}s \tag{6}$$

on the semicircular contour shown in Fig. 1a. Here, the angle ϕ is defined as:

$$\phi(\mathbf{x}, \mathbf{y}) = \arctan\left(\frac{y_2 - x_2}{y_1 - x_1}\right) \tag{7}$$

and s is the distance measured along the contour. Also, $C = C_R \cup C_0$.

It can be shown that (e.g., Ghosh et al., 1986) that, at a regular point on C, $(\mathrm{d}\phi/\mathrm{d}s) = (1/r)(\partial r/\partial n)$, where $\mathbf{n}$ is the unit outward normal to C at that point. Thus, $(\mathrm{d}\phi/\mathrm{d}s) \sim O(1/r)$ as $\mathbf{y} \rightarrow \mathbf{x}$.

It is easy to show that:

$$J = \int_{C_R} \frac{\mathrm{d}\phi}{\mathrm{d}s} \, \mathrm{d}s = \pi \tag{8}$$

$$I_{\mathrm{CPV}}(\mathbf{x} = 0) = ⨍_C \frac{\mathrm{d}\phi}{\mathrm{d}s} \, \mathrm{d}s = \pi + \lim_{\epsilon \to 0} \left[\int_A^B \frac{\mathrm{d}\phi}{\mathrm{d}s} \, \mathrm{d}s + \int_C^D \frac{\mathrm{d}\phi}{\mathrm{d}s} \, \mathrm{d}s \right] = \pi \tag{9}$$

$$I_\epsilon = \int_\pi^{2\pi} \frac{d\phi}{ds}\, ds = \pi \tag{10}$$

To calculate I_{HFP}, observe that for $\xi \notin C$:

$$\int_C \frac{d\phi}{ds}(\boldsymbol{\xi}, \mathbf{y})\, ds(\mathbf{y}) = 2\pi\gamma(\boldsymbol{\xi}) \tag{11}$$

(with $\gamma(\xi) = 1$ for ξ inside the body and $\gamma(\xi) = 0$ otherwise), so that, using Proposition 4.1 of Toh and Mukherjee (1994) and taking the limit as $\zeta \to \mathbf{x} \in C$, one has:

$$I_{HFP} = \mathop{=\!\!\!\!\!\!\int}_C \frac{d\phi}{ds}(\mathbf{x}, \mathbf{y})\, ds(\mathbf{y}) = 2\pi\gamma(\mathbf{x}) \tag{12}$$

where $\gamma(\mathbf{x}) = 1$ for $\xi \to \mathbf{x}$ from inside the body and $\gamma(\mathbf{x}) = 0$ for $\xi \to \mathbf{x}$ from outside (see, also, Lutz et al., 1992).

In this example, since ξ is inside the body, $\gamma(\mathbf{x}) = 1$, and Eq. (1) is verified.

It is very interesting to note that with $\mathbf{x} = 0$ at a corner of the contour C (Fig. 1b), one gets $I_{CPV} = \pi/2$, $I_\epsilon = 3\pi/2$ and $I_{HFP} = 2\pi$. This verifies the fact that Eq. (1) remains valid with $\mathbf{x}$ at a corner of C.

For the case in which ξ starts outside the body, an exclusion zone is used. Now, with $\mathbf{x} = 0$ a regular point on C, $I_{CPV} = \pi$, $I_\epsilon = -\pi$ and $I_{HFP} = 0$. If $\mathbf{x}$ lies at a corner as in Fig. 1b, $I_{CPV} = \pi/2$, $I_\epsilon = -\pi/2$ and $I_{HFP} = 0$.

These results are summarized in Table 2.

This example is concerned with a special integral for which Eq. (11) remains valid for a ‘‘reasonable’’ (i.e., simple, closed, piecewise continuous, counterclockwise) contour of any shape — the situation being analogous to some complex contour integrals that allow certain deformations of the contour without changing the value of the integral. It is interesting to note that, for this example, the value of I_{HFP} depends on the approach (from inside or from outside a body) in the LTB, but not on whether the boundary point is regular or not. The reverse is true for the I_{CPV}, i.e., its value does not depend on the approach in the LTB but does depend on the nature of the boundary point. Of course, in all cases, Eq. (1) remains valid.

It is important to remind the reader that the above example pertains to I_{HFP} integrals for which I_{CPV}s exist. The situation is much more complicated when one wishes to obtain I_{HFP} values of hypersingular integrals (for which, in general, I_{CPV} values do not exist), at nonsmooth points on the boundary of a body (see, for example, Mantič and París, 1995). The I_{HFP} interpretation given in Toh and Mukherjee (1994) only pertains to smooth points on the boundary of a body.

Table 2
Values of integrals in the simple example

	Regular point	Corner point
LTB from inside	$I_{HFP} = 2\pi$ $I_{CPV} = \pi$	$I_{HFP} = 2\pi$ $I_{CPV} = \pi/2$
LTB from outside	$I_{HFP} = 0$ $I_{CPV} = \pi$	$I_{HFP} = 0$ $I_{CPV} = \pi/2$

2.4. An example that involves the solid angle

Consider the segment APB (called the surface S) of the surface of a sphere ∂B in Fig. 2a. The purpose here is to find the solid angle subtended by S (a) at a point P on S (the CPV), (b) at a point p_{I} directly below P (i.e., inside the sphere) in the limit $p_{\mathrm{I}} \to P$ (the HFP with approach from inside the sphere) and (c) same as case (b) but now with p_O directly above P (i.e., the HFP with approach from outside the sphere).

The usual formula for the solid angle subtended by a surface S at a point p is:

$$\Omega(p) = \int_S \frac{\mathbf{r} \cdot \mathrm{d}\mathbf{S}}{r^3} = \int_S \frac{(\mathbf{r} \cdot \mathbf{n})\, \mathrm{d}S}{r^3} \tag{13}$$

where $\mathbf{r} \equiv \mathbf{r}(p,q)$ is the vector from p to a point q on S, r is the magnitude of $\mathbf{r}$ and $\mathbf{n}$ is the unit normal to S at q. Here, $\mathbf{n}$ is chosen to point outwards from the sphere surface ∂B.

2.4.1. The CPV integral

Let S be an open surface which is part of a closed surface ∂B. A formula for determining the solid angle, subtended by S, at a point on it (i.e., the singular integral case) has been derived in Appendix E of Mukherjee and Mukherjee (1998). This is:

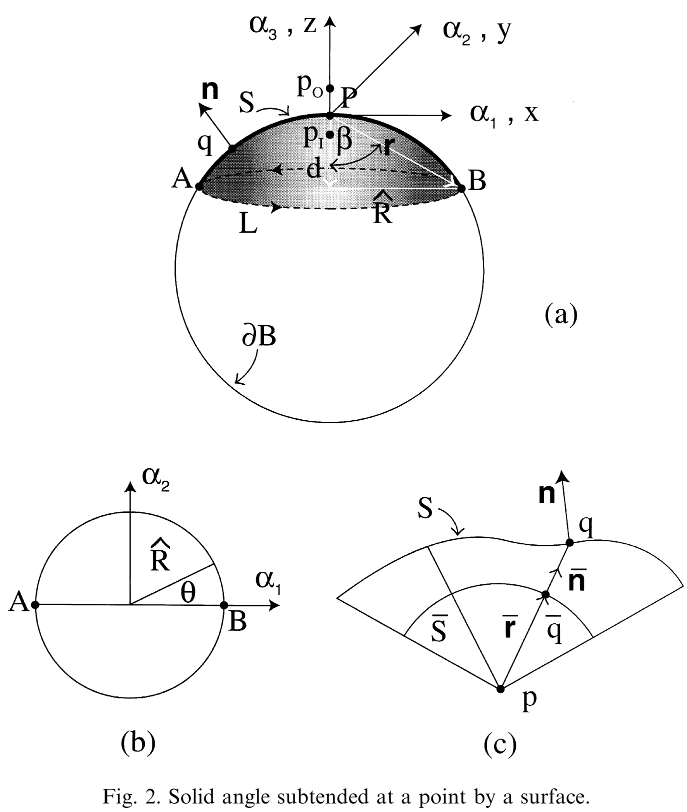

Fig. 2. Solid angle subtended at a point by a surface.

$$\Omega(P) \equiv \Omega_{\mathrm{CPV}} = \oint_L \frac{\alpha_3(\alpha_2 \mathrm{d}\alpha_1 - \alpha_1 \mathrm{d}\alpha_2)}{r(\alpha_1^2 + \alpha_2^2)} \tag{14}$$

in terms of local coordinates (α_1, α_2, α_3) at P with α_3 along the outward normal to ∂B at P and α_1 and α_2 along two mutually perpendicular tangential directions to S at P. Also, L is the bounding contour of S.

This formula is now applied to the surface S shown in Fig. 2a. With $\alpha_1 = \hat{R}\cos(\theta)$, $\alpha_2 = \hat{R}\sin(\theta)$, $\alpha_3 = -d = -\hat{R}/\tan(\beta)$ and $r = \hat{R}/\sin\beta$, one immediately has:

$$\Omega_{\mathrm{CPV}} = 2\pi\cos(\beta) \tag{15}$$

2.4.2. The HFP integrals

This time, one needs a solid angle formula for the nearly singular case when the point p is very near S but not on it. Such a formula is available in Liu (1998). A generalized version of this formula, valid for points on either side of the surface S, and also adapted to the definition of the solid angle given in Eq. 13, is:

$$\Omega(p) = \mathrm{sgn}(\bar{\mathbf{r}} \cdot \bar{\mathbf{n}}) \int_0^{2\pi} \left[1 - \cos\Phi(\theta)\right] \mathrm{d}\theta \tag{16}$$

The requirement for this formula to be valid is that one has a local coordinate system (x,y,z) at p oriented such that the positive z-axis intersects the surface S. Also, Φ is the angle between the positive z-axis and $\mathbf{r}(p,q)$ with $q \in L$, and θ is the angle between the positive x-axis and the projection of $\mathbf{r}$ in the xy plane. Finally, sgn denotes the sign of the dot product of $\bar{\mathbf{r}} \equiv \mathbf{r}(p,\bar{q})$ and $\bar{\mathbf{n}}$. Here $\bar{S}$ is part of the surface of a sphere, centered at p, radius $\bar{r}$, on which S is projected such that $q \in S \to \bar{q} \in \bar{S}$ and $\mathbf{n}$, the normal to S, $\to \bar{\mathbf{n}}$, the normal to $\bar{S}$ (Fig. 2c). Note that the vectors $\bar{\mathbf{r}}$ and $\bar{\mathbf{n}}$ are always parallel. With $\mathbf{n}$ defined as the outward normal to ∂B, they point in the same direction when p is inside ∂B and in opposite directions when p is outside ∂B. In Liu (1998), this signum function is always $+1$.

For the example shown in Fig. 2, $\Phi = \pi - \beta$, so that, taking the LTB, Eq. (16) gives:

$$\Omega(p_{\mathrm{I}}) \equiv \Omega_{\mathrm{HFPI}} = 2\pi(1 + \cos(\beta)) \tag{17}$$

Note that, for the full sphere, $\beta = 0$ and $\Omega(p_{\mathrm{I}}) = 4\pi$ as expected.

The integral $\lim_{\epsilon\to 0} I_\epsilon$ can be obtained by letting the cap APB in Fig. 2a shrink by letting $A \to P$ and $B \to P$. Now, $\beta \to \pi/2$, so that, from Eq. (17), $\lim_{\epsilon\to 0} I_\epsilon = 2\pi$. This fact is also evident by choosing a hemispherical inclusion zone S_ϵ centered at P and noting that the solid angle subtended by S_ϵ at P is 2π. Therefore, Eq. (1) remains valid.

Finally, the point p_{O} in Fig. 2a is considered. Now, the z axis points down so that $\Phi = \beta$. Also, the signum function is -1. Therefore:

$$\Omega(p_O) \equiv \Omega_{\mathrm{HFPO}} = 2\pi(\cos(\beta) - 1) \tag{18}$$

For the full sphere, $\Omega(p_O) = 0$ as expected.

This time $\lim_{\epsilon\to 0} I_\epsilon = 2\pi[\cos(\beta) - 1]_{\beta=\pi/2} = -2\pi$, a result that can also be obtained by considering a hemispherical exclusion zone centered at P. Of course, Eq. (1) is again valid.

3. Applications in BIEs of linear elasticity

It is common practice in the BIE literature (dating back to Rizzo (1967) for linear elasticity), to write a strongly singular integral, for which a finite CPV exists, in the form:

$$I = I_{\text{CPV}} + \lim_{\epsilon\to 0} I_\epsilon \tag{19}$$

One then proceeds to evaluate each of these integrals as the inclusion or exclusion zone shrinks to zero (e.g., Guiggiani and Casalini, 1987). Of particular interest here is the interpretation of the left hand side of Eq. (19) as the HFP of the integral I.

Applications of Eq. (1) in the BIEs of linear elasticity (in 2 or 3 dimensions) are presented in this section. In the interest of brevity, only LTBs from inside a body are considered here. Of course, there is no difficulty in carrying through similar arguments for LTBs from outside a body.

3.1. The (strongly singular) displacement BIE — Eq. (1) for $I = \int_{\partial B} T_{ik}(\mathbf{x}, \mathbf{y})\, ds(\mathbf{y})$

Consider (for simplicity) a simply connected body B (an open set) with boundary ∂B. The standard integral representation (e.g., Brebbia et al., 1984) for a source point $\mathbf{x}$ inside the body B (i.e., $\mathbf{x} \in B$) is (Rizzo, 1967):

$$u_i(\mathbf{x}) = \int_{\partial B} \left[U_{ik}(\mathbf{x}, \mathbf{y})\tau_k(\mathbf{y}) - T_{ik}(\mathbf{x}, \mathbf{y}) u_k(\mathbf{y}) \right] \mathrm{d}s(\mathbf{y}) \tag{20}$$

where $u_k(\mathbf{y})$ and $\tau_k(\mathbf{y})$ are components of the displacement and traction vectors, respectively, at a field point $\mathbf{y}$, and $\mathrm{d}s(\mathbf{y})$ is an infinitesimal boundary element. The usual Kelvin kernels U_{ik} and T_{ik} are available in many references (e.g., Mukherjee, 1982; Brebbia et al., 1984). These kernels, for 3-D problems, are also given in Appendix A.

The singular BIE for $\mathbf{x} \in \partial B$ can be written in several ways. A CPV form of Eq. (20) with an integral over an inclusion zone S_ϵ is (Dong and Gea, 1998)

$$u_i(\mathbf{x}) = \fint_{\partial B} \left[U_{ik}(\mathbf{x}, \mathbf{y})\tau_k(\mathbf{y}) - T_{ik}(\mathbf{x}, \mathbf{y}) u_k(\mathbf{y}) \right] \mathrm{d}s(\mathbf{y}) - u_k(\mathbf{x}) \lim_{\epsilon\to 0} \int_{S_\epsilon} T_{ik}(\mathbf{x}, \mathbf{y})\, \mathrm{d}s(\mathbf{y}) \tag{21}$$

while the well known CPV form of Eq. (20) with the corner tensor $\mathbf{C}$ is:

$$C_{ik}(\mathbf{x}) u_k(\mathbf{x}) = \fint_{\partial B} \left[U_{ik}(\mathbf{x}, \mathbf{y})\tau_k(\mathbf{y}) - T_{ik}(\mathbf{x}, \mathbf{y}) u_k(\mathbf{y}) \right] \mathrm{d}s(\mathbf{y}) \tag{22}$$

An HFP form of Eq. (20) is:

$$u_i(\mathbf{x}) = \mathop{{\int\!\!\!\!\!\!=}}\nolimits_{\partial B} \left[U_{ik}(\mathbf{x}, \mathbf{y})\tau_k(\mathbf{y}) - T_{ik}(\mathbf{x}, \mathbf{y}) u_k(\mathbf{y}) \right] \mathrm{d}s(\mathbf{y}) \tag{23}$$

while the well known regularized (weakly singular) form of Eq. (20), valid at an arbitrary point $\mathbf{x}$, is:

$$0 = \int_{\partial B} \left[U_{ik}(\mathbf{x}, \mathbf{y})\tau_k(\mathbf{y}) - T_{ik}(\mathbf{x}, \mathbf{y})(u_k(\mathbf{y}) - u_k(\mathbf{x})) \right] \mathrm{d}s(\mathbf{y}) \tag{24}$$

From Eqs. (21) and (22), one has:

$$C_{ik}(\mathbf{x}) = \delta_{ik} + \lim_{\epsilon\to 0} \int_{S_\epsilon} T_{ik}(\mathbf{x}, \mathbf{y})\, \mathrm{d}s(\mathbf{y}) \tag{25}$$

while using the rigid body mode in Eq. (22) gives:

$$I_{\mathrm{CVP}} \equiv \fint_{\partial B} T_{ik}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = -C_{ik}(\mathbf{x}) \tag{26}$$

From Eqs. (25 and 26):

$$I_{\mathrm{CVP}} + \lim{}_{\epsilon \to 0} I_\epsilon \equiv \fint_{\partial B} T_{ik}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}) + \lim{}_{\epsilon \to 0} \int_{S_\epsilon} T_{ik}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = -\delta_{ik} \tag{27}$$

Finally, use of the rigid body mode in Eq. (20) (here $\mathbf{x} \in B$) gives:

$$\delta_{ik} = -\int_{\partial B} T_{ik}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \tag{28}$$

while taking the limit of Eq. (28) as $\mathbf{x} \to \partial B$, one has (Toh and Mukherjee, 1994):

$$I_{\mathrm{HFP}} \equiv \fint\!\!\!\!=_{\partial B} T_{ik}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = -\delta_{ik} \tag{29}$$

From Eqs. (27) and (29), one immediately gets Eq. (1).

3.2. The (hypersingular) stress BIE — Eq. (1) for $I = \int_{\partial B} I_{ijkl}(\boldsymbol{x}, \boldsymbol{y}) ds(\boldsymbol{y})$

This section follows the pattern of the previous one. The stress BIE for $\mathbf{x} \in B$ is (Cruse, 1969)

$$\sigma_{ij}(\mathbf{x}) = \int_{\partial B} \left[D_{ijk}(\mathbf{x}, \mathbf{y}) \tau_k(\mathbf{y}) - S_{ijk}(\mathbf{x}, \mathbf{y}) u_k(\mathbf{y}) \right] \mathrm{d}s(\mathbf{y}) \tag{30}$$

where σ_{ij} are stress components and the kernels D_{ijk} and S_{ijk} are given elsewhere (e.g., Cruse, 1969; Mukherjee et al., 1999), and also in Appendix A (for 3-D problems).

Use of the well known rigid body mode allows one to write Eq. (30) as (Cruse and Van Buren, 1971):

$$\sigma_{ij}(\mathbf{x}) = \int_{\partial B} \left[D_{ijk}(\mathbf{x}, \mathbf{y}) \tau_k(\mathbf{y}) - S_{ijk}(\mathbf{x}, \mathbf{y}) (u_k(\mathbf{y}) - u_k(\mathbf{x})) \right] \mathrm{d}s(\mathbf{y}) \tag{31}$$

The stress HBIE for $\mathbf{x} \in \partial B$ can be written in several ways. A partially regularized CPV version of Eq. (31) with an integral over an inclusion zone is (Dong and Gea, 1998):

$$\sigma_{ij}(\mathbf{x}) = \fint_{\partial B} \left[D_{ijk}(\mathbf{x}, \mathbf{y}) \tau_k(\mathbf{y}) - S_{ijk}(\mathbf{x}, \mathbf{y}) (u_k(\mathbf{y}) - u_k(\mathbf{x})) \right] \mathrm{d}s(\mathbf{y}) + u_{k,\,l}(\mathbf{x}) \lim{}_{\epsilon \to 0} \int_{S_\epsilon} \left[E_{klmh} D_{ijm}(\mathbf{x}, \mathbf{y}) n_h(\mathbf{x}) - S_{ijk}(\mathbf{x}, \mathbf{y}) (y_l - x_l) \right] \mathrm{d}s(\mathbf{y}) \tag{32}$$

Here $\mathbf{E}$ is the usual elasticity tensor.

The second term on the right-hand side of Eq. (32) is the contribution of the integral in Eq. (31) across a singularity as $\mathbf{y} \to \mathbf{x}$. It is obtained by expanding the displacement and traction in Taylor series as:

$$u_k(\mathbf{y}) = u_k(\mathbf{x}) + u_{k,\,l}(\mathbf{x})(y_l - x_l) + O(r^2) \tag{33}$$

$$\tau_m(\mathbf{y}) = \tau_m(\mathbf{x}) + O(r) = E_{klmh} u_{k,\,l}(\mathbf{x}) n_h(\mathbf{x}) + O(r) \tag{34}$$

A CPV version of Eq. (31) with a free term, but without an integral over an inclusion zone, is:

$$\Lambda_{ijkl}(\mathbf{x})u_{k,\,l}(\mathbf{x}) = \fint_{\partial B} \left[D_{ijk}(\mathbf{x},\,\mathbf{y})\tau_k(\mathbf{y}) - S_{ijk}(\mathbf{x},\,\mathbf{y})(u_k(\mathbf{y}) - u_k(\mathbf{X}))\right] \mathrm{d}s(\mathbf{y}) \tag{35}$$

while an HFP form of Eq. (30) is:

$$\sigma_{ij}(\mathbf{x}) = \bbint_{\partial B} \left[D_{ijk}(\mathbf{x},\,\mathbf{y})\tau_k(\mathbf{y}) - S_{ijk}(\mathbf{x},\,\mathbf{y})u_k(\mathbf{y})\right] \mathrm{d}s(\mathbf{y}) \tag{36}$$

Finally, a regularized (regular in 2-D, weakly singular in 3-D) form of Eq. (36) can be obtained by using both the rigid body and linear displacement modes in Eq. (30) (Rudolphi, 1991; Lutz et al., 1992). (The linear mode is given by Eqs. (33 and 34) without the $O(r^2)$ and $O(r)$ terms in these equations). The result is:

$$0 = \int_{\partial B} \left[D_{ijk}(\mathbf{x},\,\mathbf{y})(\sigma_{kh}(\mathbf{y}) - \sigma_{kh}(\mathbf{x}))n_h(\mathbf{y}) - S_{ijk}(\mathbf{x},\,\mathbf{y})(u_k(\mathbf{y}) - u_k(\mathbf{x}) - u_{k,\,l}(\mathbf{x})(y_l - x_l))\right] \mathrm{d}s(\mathbf{y}) \tag{37}$$

Cruse and Richardson (1996) have proved that Eq. (37) is valid at an arbitrary source point $\mathbf{x}$ provided that the displacement and stress fields satisfy certain conditions. For a detailed discussion of smoothness requirements and relaxation strategies for singular and hypersingular integral equations, the reader is referred to Martin et al. (1998).

From Eqs. (32) and (35), one has:

$$\Lambda_{ijkl}(\mathbf{x}) = E_{ijkl} - \lim_{\epsilon \to 0} \int_{S_\epsilon} I_{ijkl}(\mathbf{x},\,\mathbf{y})\, \mathrm{d}s(\mathbf{y}) \tag{38}$$

where:

$$I_{ijkl}(\mathbf{x},\,\mathbf{y}) = E_{klmh} D_{ijm}(\mathbf{x},\,\mathbf{y})n_k(\mathbf{x}) - S_{ijk}(\mathbf{x},\,\mathbf{y})(y_l - x_l) \tag{39}$$

Also, by using the linear displacement mode in Eq. (35), one can show that:

$$\Lambda_{ijkl}(\mathbf{x}) = \fint_{\partial B} I_{ijkl}(\mathbf{x},\,\mathbf{y})\, \mathrm{d}s(\mathbf{y}) \tag{40}$$

From Eqs. (38) and (40):

$$E_{ijkl} = \fint_{\partial B} I_{ijkl}(\mathbf{x},\,\mathbf{y})\, \mathrm{d}s(\mathbf{y}) + \lim_{\epsilon \to 0} \int_{S_\epsilon} I_{ijkl}(\mathbf{x},\,\mathbf{y})\, \mathrm{d}s(\mathbf{y}) \tag{41}$$

Finally, use of the linear displacement mode in Eq. (31) gives, for $\mathbf{x} \in B$,

$$E_{ijkl} = \int_{\partial B} I_{ijkl}(\mathbf{x},\,\mathbf{y})\, \mathrm{d}s(\mathbf{y}) \tag{42}$$

and taking the limit as $\mathbf{x} \to \partial B$ (Toh and Mukherjee, 1994), one gets:

$$E_{ijkl} = \bbint_{\partial B} I_{ijkl}(\mathbf{x},\,\mathbf{y})\, \mathrm{d}s(\mathbf{y}) \tag{43}$$

A comparison of Eqs. (41) and (43) verifies the validity of Eq. (1) in this case as well.

Finally, (as an aside), it is observed that the regularized BIE Eq. (24) relates $\mathbf{u}$ and τ on ∂B. In effect, so does the regularized HBIE Eq. (37) since the displacement gradient and stress at $\mathbf{x}$ (for $\mathbf{x} \in \partial B$) can be obtained in terms of the traction and tangential displacement derivatives at that point (Cruse and Van

Buren, 1971; Sladek and Sladek, 1986; Lutz et al., 1992; Mukherjee et al., 1999). It is noted that Eqs. (24) and (37) must be dependent when collocated at the same point $\mathbf{x} \in \partial B$. Otherwise, one would be able to solve an elasticity problem without the need for any boundary conditions!

Appendix A

BIE and HBIE kernels

The Kelvin kernels for the displacement BIE Eq. (20) for 3-D linear elasticity are:

$$U_{ik} = \frac{1}{16\pi G(1-\nu)r}\left[(3-4\nu)\delta_{ik} + r_{,i} r_{,k}\right]$$

$$T_{ik} = -\frac{1}{8\pi(1-\nu)r^2}\left[\{(1-2\nu)\delta_{ik} + 3r_{,i} r_{,k}\}\frac{\partial r}{\partial n} + (1-2\nu)(r_{,k} n_i - r_{,i} n_k)\right]$$

The corresponding kernels for the stress BIE Eq. (30) are:

$$D_{ijk} = E_{ijmn}\frac{\partial U_{mk}}{\partial x_n} = \frac{1}{8\pi(1-\nu)r^2}\left[(1-2\nu)(\delta_{ik} r_{,j} + \delta_{kj} r_{,i} - \delta_{ji} r_{,k}) + 3r_{,i} r_{,j} r_{,k}\right]$$

$$S_{ijk} = E_{ijmn}\frac{\partial T_{mk}}{\partial x_n} = \frac{G}{4\pi(1-\nu)r^3}\Bigg[\{(1-2\nu)\delta_{ij} r_{,k} + \nu(\delta_{ik} r_{,j} + \delta_{jk} r_{,i}) - 5r_{,i} r_{,j} r_{,k}\}3\frac{\partial r}{\partial n} + 3\nu(n_i r_{,j} r_{,k} + n_j r_{,k} r_{,i}) - (1-4\nu)n_k\delta_{ij} + (1-2\nu)(3n_k r_{,i} r_{,j} + n_j\delta_{ki} + n_i\delta_{jk}\Bigg]$$

In the above, r is the distance between a source point $\mathbf{x}$ and a field point $\mathbf{y}$, G and ν are the shear modulus and Poisson's ratio, respectively, δ_{ik} are the components of the Kronecker delta and, $k \equiv \partial/\partial y_k$. Also, the components of the normal as well as the normal derivative $(\partial r/\partial n)$ are evaluated at the field point $\mathbf{y}$.

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